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Interesting relations in Fock space*

by

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ABSTRACT

Certain non-linear relations between the generators of the (q-deformed) Heisenberg algebra are found. Some of these relations are invariant under quantization and q-deformation.

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In this Talk I want to present different relations appearing in the Fock space generated by the q-deformed Heisenberg algebra. In a certain particular case, some of these relations can be summarized by the theorem:

(main) THEOREM. The following differential identities hold for any natural number n and $\delta \in \mathbf{C}$

$$\left(x^2\frac{d}{dx} - nx\right)^{n+1} = x^{2n+2}\frac{d^{n+1}}{dx^{n+1}} \ , \quad \left(\frac{d^2}{dx^2}x - n\frac{d}{dx}\right)^{n+1} = \frac{d^{2n+2}}{dx^{2n+2}}x^{n+1} \ , \tag{1}$$

$$\left(x\frac{d}{dx}x\right)^n = x^n\frac{d^n}{dx^n}x^n \ , \quad \left(\frac{d}{dx}x\frac{d}{dx}\right)^n = \frac{d^n}{dx^n}x^n\frac{d^n}{dx^n} \ , \tag{2}$$

$$x\frac{d}{dx}\left(x\frac{d}{dx} - 1\right)\dots\left(x\frac{d}{dx} - n\right) = x^{n+1}\frac{d^{n+1}}{dx^{n+1}} \ , \tag{3}$$

$$\prod_{k=0}^{n}\left[x\left(1 - e^{-\delta\frac{d}{dx}}\right) - k\delta\right] = x^{(n+1)}\left(1 - e^{-\delta\frac{d}{dx}}\right)^{n+1} \ , \tag{4}$$

where $x^{(n+1)} = x(x-\delta)(x-2\delta)\dots(x-n\delta)$.

The proof can be carried out by induction. Later we will show that these relations can be easily generalized to various Fock spaces.

Take some 3-dimensional complex algebra with an identity operator and with two elements a and b, obeying the relation

$$ab - qba = p \ , \tag{5}$$

where p, q are any complex numbers. The algebra with the identity operator generated by the elements a, b and obeying (1) is usually called the q-deformed Heisenberg algebra h_q (See, for example, papers [1] and references and a discussion therein). The parameter q is called the parameter of quantum deformation. If $p = 0$, then (5) describes the non-commutative (quantum) plane (See, for example, [2]), while if $q = 1$, then h_q becomes the ordinary (classical) Heisenberg algebra and the parameter p plays a role of the Planck constant (See, for example, [3]). The operator linear space of all holomorphic functions in a, b with vacuum

$$a|0> = 0$$

is called a *Fock* space. Then the following theorem holds:

THEOREM 1 [4]. For any $p, q \in \mathbf{C}$ in (5) the following identities are true

$$(aba)^n \ = \ a^n b^n a^n, \quad n = 1, 2, 3, \dots \ , \tag{6.1}$$

and, if $q \neq 0$, also

$$(bab)^n \ = \ b^n a^n b^n, \quad n = 1, 2, 3, \dots \ . \tag{6.2}$$

The proof can be carried out by induction using the following easy lemma

LEMMA 1. For any $p, q \in \mathbf{C}$ in (5),

$$ab^n \ - \ q^n b^n a \ = \ p\{n\} b^{n-1} \ , \tag{7.1}$$

$$a^n b \ - \ q^n b a^n \ = \ p\{n\} a^{n-1} \ , \tag{7.2}$$

where $n = 1, 2, 3 \dots$ and $\{n\} = \frac{1-q^n}{1-q}$ is the so-called q-number (See, for example, [5]).

Now let us proceed to the proof of (6.1). For $n = 1$ relation (6.1) holds trivially. Assume that (6.1) holds for some n and check that it holds for $n+1$. It is easy to write the chain of equalities :

$$(aba)^{n+1} \ = \ (aba)^n aba \ = \ a^n b^n a^n aba \ = \ a^n (b^n a)(a^n b) a \ =$$

$$= \ a^n \left(\frac{ab^n}{q^n} - \frac{p}{q^n}\{n\} b^{n-1}\right)(q^n b a^n + p\{n\} a^{n-1}) a \ = \ a^{n+1} b^{n+1} a^{n+1} \ , \tag{$\diamondsuit$}$$

where the relations (7.1) and (7.2) were used in this chain. In an analogous way, one can prove (6.2). *q.e.d.*

Theorem 1 leads to two corollaries, both of them easily verified:

Corollary 1. For any $p, q \in \mathbf{C}$ in (5) and natural numbers n, k,

$$(\underbrace{ababa....aba}_{2k+1})^n = \underbrace{a^n b^n a^n ... b^n a^n}_{2k+1}, \quad n, k = 1, 2, 3, \dots \tag{8}$$

Corollary 2. Let $T_k^{(n)} = \underbrace{a^n b^n a^n \dots b^n a^n}_{2k+1}$. Then, for any $p, q \in \mathbf{C}$ in (5), the following relation holds

$$[T_k^{(n)}, T_k^{(m)}] \ = \ 0 \ , \tag{9.1}$$

as well as the more general relation:

$$[T_k^{(n_1)}T_k^{(n_2)}\dots T_k^{(n_i)}, T_k^{(m_1)}T_k^{(m_2)}\dots T_k^{(m_j)}] \;=\; 0 \;, \tag{9.2}$$

where $[\alpha,\beta] \equiv \alpha\beta - \beta\alpha$ is the standard commutator and $\langle n\rangle, \langle m\rangle$ are sets of any non-negative, integer numbers.

It is evident that formulas (8), (9.1) and (9.2) remain correct under the replacement $a \rightleftharpoons b$, if $q \neq 0$. It is worth noting that the algebra h_q under an appropriate choice of the parameters p, q has a natural representation

$$a = x \;,\; b = D \qquad (\star)$$

where the operator $Df(x) = \frac{f(x)-f(qx)}{x(1-q)}$ is the dilatationally-invariant shift operator and it is usually called the Jackson symbol (See, for example, [5]). The relation (6.1) then becomes

$$(xDx)^n \;=\; x^n D^n x^n \;,$$

while relation (6.2) becomes

$$(DxD)^n \;=\; D^n x^n D^n \;.$$

Since for $q \to 1$, the operator $D \to \frac{d}{dx}$, the above relations become differential identities (2).

THEOREM 2 [6]. For any $p, q \in \mathbf{C}$ in (5) and natural n, m the following identities hold:

$$[a^n b^n, a^m b^m] \;=\; 0 \;, \tag{10.1}$$

$$[a^n b^n, b^m a^m] \;=\; 0 \;, \tag{10.2}$$

$$[b^n a^n, b^m a^m] \;=\; 0 \;. \tag{10.3}$$

The proof is based on the following easy lemma

LEMMA 2. For any $p, q \in \mathbf{C}$ in (5) and natural number n

$$a^n b^n \;=\; P(ab) \;, \tag{11.1}$$

$$b^n a^n \;=\; Q(ab)\;, \tag{11.2}$$

where P, Q are some polynomials in one variable of order not higher than n.

Let us introduce a notation $t^{(n)} = a^n b^n$ or $b^n a^n$, in which the order of the multipliers is not essential. The statement of the Theorem 2 can now be written as $[t^{(n)}, t^{(m)}] = 0$ and the following is true;

Corollary 3. The commutator

$$[t^{(n_1)}t^{(n_2)}\ldots t^{(n_i)},\; t^{(m_1)}t^{(m_2)}\ldots t^{(m_j)}] \;=\; 0 \tag{12}$$

holds for any $p, q \in \mathbf{C}$ in (5) and any sets $\langle n\rangle, \langle m\rangle$ of non-negative, integer numbers.

One can make sense of (6.1) and (6.2), (8), (9.1) and (9.2), (10.1)-(10.3), (12) as follows: In the algebra of polynomials in a, b there exist relations invariant under a variation of the parameters p, q in (5). Also formulas (6.1) and (6.2), (8) can be interpreted as formulas of a certain special ordering other than the standard lexicographical one.

A natural question can be raised: Is the existence of the relations (6.1) and (6.2), (8), (9.1) and (9.2), (10.1)-(10.3), (12) connected unambiguously to the algebra h_q, or are there more general algebra(s) leading to those relations? Some answer is given by the following theorem

THEOREM 3 [6]. If two elements a, b of a Banach algebra with unit element are related by

$$ba = f(ab)\;, \tag{13}$$

where f is a holomorphic function in a neighbourhood of Spec $\{ab\}\cup$Spec $\{ba\}$, then relations (6.1), (8), (10.3) hold. If, in addition, the function f is single-sheeted, then (10.2) also holds.

The proof is essentially based on the fact that, if the function f in (13) is holomorphic, then for *any* holomorphic F

$$bF(ab) \;=\; F(ba)b\;, \tag{14}$$

which guarantees the correctness of the statement (11.2) of Lemma 2, although Q is no longer polynomial. This immediately proves (10.3). The

relation (6.1) can be proved by induction and an analogue of the logical chain ($\diamondsuit$) is

$$(aba)^{n+1} \;=\; (aba)^n aba \;=\; a^n b^n a^n aba \;=\; a^n(b^n a^n)aba \;=$$
$$=\; [b^n a^n = Q(ab), \text{see Lemma 2}] \;=$$
$$=\; a^n Q(ab)(ab)a \;=\; a^n(ab)Q(ab)a \;=\; a^{n+1}b^{n+1}a^{n+1}\;.$$

An extra condition that f be single-sheeted implies that $ab = f^{-1}(ba)$, which immediately leads to the statement (10.2). *q.e.d.*

It is evident that the replacement $a \rightleftharpoons b$ in Theorem 3 leads to the fulfilment of the equalities (6.2), (10.1) and (10.2) as well.

There exists another type of relations in Fock space stemming from the fact that some algebras are contained in the Fock space and these algebras can possess finite-dimensional representations. One can prove the theorem:

THEOREM 4. For any $p, q \in \mathbf{C}$ in (5) and $n = 1, 2, 3, \ldots$ the following identities are true

$$(b^2a - \{n\}b)^{n+1} = q^{n(n+1)}b^{2n+2}a^{n+1}\;, \tag{15.1}$$

and, if $q \neq 0$, also

$$(ba^2 - \{n\}a)^{n+1} = q^{n(n+1)}b^{n+1}a^{2n+2}\;, \tag{15.2}$$

where $\{n\} = \frac{1-q^n}{1-q}$ is the q-number.

In order to prove this theorem, we need, at first, to state the following observation

LEMMA 3 [7]. For any $p, q \in \mathbf{C}$ in (5) and $\alpha \in \mathbf{C}$, three elements of the Fock space

$$\begin{aligned} j^+ \;&=\; b^2a - \{\alpha\}b\;, \\ j^0 \;&=\; ba - \frac{\{\alpha\}\{\alpha+1\}}{\{2\alpha+2\}}\;, \\ j^- \;&=\; a\;, \end{aligned} \tag{16}$$

(modified by some multiplicative factors) obey q-deformed commutation relations

$$
\begin{aligned}
q\tilde{j}^0\tilde{j}^- \ - \ \tilde{j}^-\tilde{j}^0 \ &= \ -\tilde{j}^- \ , \\
q^2\tilde{j}^+\tilde{j}^- \ - \ \tilde{j}^-\tilde{j}^+ \ &= \ -(q+1)\tilde{j}^0 \ , \\
\tilde{j}^0\tilde{j}^+ \ - \ q\tilde{j}^+\tilde{j}^0 \ &= \ \tilde{j}^+ \ ,
\end{aligned}
\tag{17}
$$

forming the algebra $s\ell_{2q}$. If $q \to 1$, these commutation relations become the standard ones for $s\ell_2$. If $\alpha = n$ is a non-negative integer number, the generators (16) form the finite-dimensional representation corresponding to the (operator) finite-dimensional representation space

$$
V_n =< 1, b, b^2, \ldots, b^n > \tag{18}
$$

in the Fock space.

Validity of this Lemma can be checked by direct calculation. The proof of Theorem 4 is based on an evident fact [8] that a positive-root (negative-root) generator in finite-dimensional representation taken in the power of the dimension of the finite-dimensional representation annihilates the space of the finite-dimensional representation and, correspondingly, acts in its complement. The operator $(j^+)^{n+1}$ at $\alpha = n$ annihilates (18) and hence it must be proportional (from the right) to a^{n+1}. Since j^+ is graded with the grading equals to (-1), $(j^+)^{n+1}$ must be proportional (from the left) to b^{2n+2}. What is left in this consideration is a value of possible multiplicative factor appearing in r.h.s. (15.1). This factor is equal to $q^{n(n+1)}$ and can be found by direct calculation. Equation (15.2) can be proved analogously, with only minor modifications.

THEOREM 5 [4]. For any $p, q \in \mathbf{C}$ in (5) and $n = 1, 2, 3, \ldots$ the following identity holds :

$$
ba(ba - \{1\}) \ldots (ba - \{n\}) \ = \ q^{\frac{n(n+1)}{2}} b^{n+1} a^{n+1} \tag{19}
$$

The proof can be carried out by induction. Now let us consider how the identity (19) looks for different representations of the algebra (5).

Taking the representation ($\star$) for the algebra (5) and plugging it into (19), we arrive at an identity for the dilatationally-invariant shift operator D :

$$
xD(xD - \{1\}) \ldots (xD - \{n\}) \ = \ q^{\frac{n(n+1)}{2}} x^{n+1} D^{n+1} \tag{20}
$$

If $q \to 1$ the representation $(\star)$ degenerates into

$$a = \frac{d}{dx} \ , \ b = x \qquad (\star\star)$$

and (19) coincides to the identity (3). Recently a more general representation of the algebra (5) was found then $(\star\star)$ at $q = 1$ is characterized by a free parameter $\delta \in C$ [9]:

$$a = \frac{(e^{\delta \frac{d}{dx}} - 1)}{\delta} \ , \ b = xe^{-\delta \frac{d}{dx}} \ , \tag{21.1}$$

or,

$$a \ = \ \mathcal{D}_+ \ , \ b \ = \ x(1 - \delta \mathcal{D}_-) \ , \tag{21.2}$$

where $\mathcal{D}_\pm f(x) \ = \ \frac{f(x \pm \delta) - f(x)}{\pm \delta}$ are the (translationally-invariant) finite-difference operators. After substitution of (21.1) in (19) for $q = 1$ a simple transformation the differential identity (4) appears. This identity takes a slightly different form if the realization (21.2) is used :

$$xD_-(xD_- - 1) \ldots (xD_- - n) \ = \ \delta^{n+1} x^{(n+1)} D_-^{n+1} \tag{22}$$

(cf.(20)).

Now take an algebra generated by three generators

$$ab - pba = F(N) \ , \ qNa - aN = -a \ , \ Nb - qbN = b \ , \tag{23}$$

(cf.(5)), where $p, q \in C$, F a holomorphic function in a neighbourhood of $Spec\{N\} \cup \{0\}$.

THEOREM 6. For any $p, q \neq 0 \in \mathbf{C}$ in (23) the identities (6.1), (6.2) and (10.1) – (10.3) hold, and also

$$ba \left(ba - F\left(\frac{N-1}{q}\right)\right) \ldots \left(ba - \sum_{k=1}^{n} p^{k-1} F\left(\frac{N - \{k\}}{q^k}\right)\right) \ = \ p^{\frac{n(n+1)}{2}} b^{n+1} a^{n+1} \ . \tag{24}$$

It is worth mentioning that for certain cases the formula (24) was known: $p = 1$ [4], $p = q$ [10].

The proof is carried out by induction and is based on a certain generalization of Lemma 2 :

LEMMA 4. For any $p, q \neq 0 \in \mathbf{C}$ in (23) and n a natural number

$$a^n b^n \;=\; \sum_{k=0}^{n} \alpha_k(N)(ab)^k \;, \tag{25.1}$$

$$b^n a^n \;=\; \sum_{k=0}^{n} \beta_k(N)(ab)^k \;, \tag{25.2}$$

where α, β are calculable functions.

We also use the simple observation that in (23)

$$af(N) = f(qN+1)a \quad , \quad f(N)b = bf(qN+1) \;,$$

$$[f(N), a^n b^n] \;=\; [f(N), b^n a^n] \;=\; 0 \;,\; n = 1, 2, \dots \;,$$

where $f(N)$ is a holomorphic function in a neighbourhood of $\mathit{Spec}\{N\} \cup \{\mathit{0}\}$.

It is worth noting that the algebra (5) or (13) can be interpreted as a deformation of the Heisenberg algebra. In turn, the algebra (23) contains as special cases:

(i) sl_2 ($p, q = 1, F = 2N$),

(ii) the q-deformed algebra sl_{2q} ($p = q^2, F = (q+1)N$, see (17)),

(iii) the quantum group $U_q(sl_2)$ ($p, q = 1, F = \frac{\sin \tilde{q}N}{\tilde{q} - \tilde{q}^{-1}}$), and even,

(iv) the Heisenberg algebra ($q = 1, F = 1$) as a sub-algebra, and

(v) the q-deformed Heisenberg algebra ($F = 1$) as a sub-algebra.

All the above demonstrates the general nature of the invariant identities (6.1), (6.2) and (10.1) – (10.3).

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